Signals are converted into frequency domain

i.e time domain ... frequency domainge convert ...

To represent the signal in frequency domain

main intention of converting time domain to frequency domain

Fourier Transforms

Infinite fourier transforms and Inverse fourier transforms

The infinite fourier transforms or simply the fourier transform of a real valued function $f(x)$ is defined by $F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{iux}\, dx$

on integration we obtain a f^n of u which is usually denoted by $F(u)$ or $\hat{f}(u)$

The inverse fourier transform of $F(u)$ denoted by $F^{-1}[F(u)]$ or $F^{-1}[\hat{f}(u)]$ is defined by the integral $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u) e^{-iux}\, du$

on integration we obtain a f^n of x

$f(x) = F^{-1}[F(u)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u) e^{-iux}\, du$

Fourier cosine and fourier sine transforms
Inverse fourier cosine & I.F. sine transforms

If $f(x)$ is defined for all possible values of x, we define the following

TYPE	TRANSFORM	INVERSE TRANSFORM
Fourier transform	$\int_{-\infty}^{\infty} f(x)e^{iux}dx = F(u)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u) e^{-iux} du = f(x)$
Fourier cosine transform	$\int_0^{\infty} f(x)\cos ux\, dx = F_c(u)$	$\frac{2}{\pi}\int_0^{\infty} F_c(u)\cos ux\, du = f(x)$
Fourier sine transform	$\int_0^{\infty} f(x)\sin ux\, dx = F_s(u)$	$\frac{2}{\pi}\int_0^{\infty} F_s(u)\sin ux\, du = f(x)$

Properties

① Linearity property:

$F_c[c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)]$

$= c_1 F_c[f_1(x)] + c_2 F_c[f_2(x)] + \cdots + c_n F_c[f_n(x)]$

② change of scale property:

If $F_c[f(x)] = F_c(u)$ then

$F_c[f(ax)] = \frac{1}{a} F_c(u/a)$

③ Modulation properties

If $F_s[f(x)] = F_s(u)$ and $F_c[f(x)] = F_c(u)$ then

① $F_s[f(x)\cos ax] = \frac{1}{2}[F_s(u+a) + F_s(u-a)]$

② $F_s[f(x)\sin ax] = \frac{1}{2}[F_c(u-a) - F_c(u+a)]$

③ $F_c[f(x)\cos ax] = \frac{1}{2}[F_c(u+a) + F_c(u-a)]$

④ $F_c[f(x)\sin ax] = \frac{1}{2}[F_s(u+a) - F_s(u-a)]$

Problems

① Find the Complex fourier transform of the function $f(x) = \begin{cases} 1 & \text{for } |x| \le a \\ 0 & \text{for } |x| > a \end{cases}$

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and hence evaluate $\int_0^{\infty} \frac{\sin x}{x}\, dx$

Solⁿ: Fourier transform of $f(x)$ is given by

$$F(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

$$f(x) = \begin{cases} 1 & \text{for } -a \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

$$= \int_{-a}^{a} 1 \cdot e^{iux}\, dx$$

$$= \left[\frac{e^{iux}}{iu}\right]_{-a}^{a}$$

$$= \frac{1}{iu}\left[e^{iua} - e^{-iua}\right]$$

w.K.T $e^{i\theta} = \cos\theta + i\sin\theta$

$$= \frac{1}{iu}\left[(\cancel{\cos au} + i\sin au) - (\cancel{\cos au} - i\sin au)\right]$$

$$= \frac{1}{iu}\, 2i\sin au$$

$$\boxed{F(u) = \frac{2\sin au}{u}}$$

Let us evaluate $\int_0^\infty \frac{\sin x}{x}\, dx$

$$F(u) = \frac{2\sin au}{u}$$

$$F(-u) = +\frac{2\sin a(-u)}{-u}$$

$$= \frac{-2\sin au}{-u} = \frac{2\sin au}{u} = F(u)$$

$F(-u) = F(u)$

$\therefore$ $F(u)$ is an even function

Inverse Fourier transform is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du = f(x)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin au}{u}\, e^{-iux}\, du = f(x)$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin au}{u}\, e^{-iux}\, du = f(x)$$

$f(x) = 1$ for $|x| \leq a$

Now, let us take $x = 0$

value of $f(x)$ at $x = 0$ is 1

i.e $f(0) = 1$

$$f(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin au}{u}\, e^{0}\, du$$

$$1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin au}{u}\, du$$

$\frac{\sin au}{u}$ is even function

$$\pi = 2\int_{0}^{\infty} \frac{\sin au}{u}\, du$$

$$\int_{0}^{\infty} \frac{\sin au}{u}\, du = \frac{\pi}{2}$$

put $a=1$ and changing u to x

$$\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \pi/2$$

② If $f(x) = \begin{cases} 1-x^2 & |x|<1 \\ 0 & |x| \geq 1 \end{cases}$

Find the fourier transform of $f(x)$ & hence find the Value of

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① $\int_0^{\infty} \frac{x\cos x - \sin x}{x^3} dx$

② $\int_0^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx$

Solⁿ:

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{iux} dx$$

$$f(x) = \begin{cases} 1-x^2, & \text{for } -1<x<1 \\ 0 & \text{otherwise} \end{cases}$$

$$F(u) = \int_{-1}^{1} (1-x^2) e^{iux} dx$$

Apply Bernoulli's rule

$$F(u) = \left[(1-x^2)\frac{e^{iux}}{iu} - (-2x)\frac{e^{iux}}{i^2u^2} - 2\frac{e^{iux}}{i^3u^3} \right]_{-1}^{1}$$

$$= \left[(1-x^2)\frac{e^{iux}}{iu} + 2x\frac{e^{iux}}{-u^2} - \frac{2e^{iux}}{-iu^3} \right]_{-1}^{1}$$

$$i^2 = -1, \quad \frac{1}{i} = -i$$

$$= \left[(1-x^2)\frac{e^{iux}}{u} \times -i - \frac{2xe^{iux}}{u^2} - \frac{i2e^{iux}}{u^3} \right]_{-1}^{1}$$

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$$= \left[\left\{ 0 - \frac{2e^{iu}}{u^2} - \frac{2ie^{iu}}{u^3} \right\} - \left\{ 0 + \frac{2e^{-iu}}{u^2} - \frac{2ie^{-iu}}{u^3} \right\} \right]$$

$$= \frac{-2e^{iu}}{u^2} - \frac{2ie^{iu}}{u^3} - \frac{2e^{-iu}}{u^2} + \frac{2ie^{-iu}}{u^3}$$

$$= \frac{-2}{u^2}(e^{iu} + e^{-iu}) - \frac{2i}{u^3}(e^{iu} - e^{-iu})$$

$$= \frac{-2}{u^2}\left[(\cos u + \cancel{i\sin u}) + (\cos u - \cancel{i\sin u})\right]$$

$$- \frac{2i}{u^3}\left[(\cancel{\cos u} + i\sin u) - (\cancel{\cos u} - i\sin u)\right]$$

$$= \frac{-2}{u^2}\left[2\cos u\right] + \left[2i\sin u\right] \times -\frac{2i}{u^3}$$

$$= \frac{-4}{u^2}\left[\cos u\right] - \frac{4(-1)\sin u}{u^3}$$

$$= \frac{-4u\cos u + 4\sin u}{u^3}$$

$$F(u) = \frac{4(\sin u - u\cos u)}{u^3}$$

Let us evaluate $\int_0^\infty \frac{-\cancel{\sin x} + x\cos x}{x^3}\,dx$

$$F(u) = \frac{4(\sin u - u\cos u)}{u^3}$$

$$F(-u) = \frac{4(-\sin u + u\cos u)}{-u^3}$$

$$= \frac{+4(\sin u - u\cos u)}{+u^3}$$

$F(-u) = F(u)$ is a even function

Inverse fourier transform is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du \quad = f(x)$$

$f(x) = 1-x^2$

put $x=0$, $f(0)=1$ & use the expression of $F(u)$

$$f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4(\sin u - u\cos u)}{u^3}\, e^{0}\, du$$

$$1 = \frac{1}{2\pi} \times 8 \int_{0}^{\infty} \frac{\sin u - u\cos u}{u^3}\, du$$

$$1 = \frac{4}{\pi}\int_{0}^{\infty} \frac{\sin u - u\cos u}{u^3}\, du$$

$$\int_{0}^{\infty} \frac{\sin u - u\cos u}{u^3}\, du = +\pi/4$$

$$-\int_{0}^{\infty} \frac{u\cos u - \sin u}{u^3}\, du = \pi/4$$

$$\int_{0}^{\infty} \frac{u\cos u - \sin u}{u^3}\, du = -\pi/4$$

changing u to x

$$\int_{0}^{\infty} \frac{x\cos x - \sin x}{x^3}\, dx = -\pi/4$$

(b) $\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx$

$$F(u) = \frac{4(\sin u - u\cos u)}{u^3}$$

$F(-u) = F(u)$, is a even function

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4(\sin u - u\cos u)}{u^3} e^{-iux}\, du$$

put $x = 1/2$

$$f(x) = 1 - x^2 = 1 - (1/2)^2 = 1 - 1/4 = 3/4$$

$$\frac{3}{4} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4(\sin u - u\cos u)}{u^3} e^{-iu \cdot 1/2}\, du$$

$$\frac{3}{4} = \frac{4}{2\pi}\int_{-\infty}^{\infty} \frac{\sin u - u\cos u}{u^3}\left(\cos u/2 - i\sin u/2\right) du$$

$F(u) = \frac{\sin u - u\cos u}{u^3}$ is even function

$$\frac{3}{4} = \frac{\cancel{2\pi}}{\pi} \times 2\int_0^\infty \frac{\sin u - u\cos u}{u^3}\left(\cos u/2 - i\sin u/2\right) du$$

$$\frac{3\pi}{16} = \int_0^\infty \frac{(\sin u - u\cos u)}{u^3}\left(\cos u/2 - i\sin u/2\right) du$$

equating real parts on B.S

$$3\pi/16 = \int_0^\infty \frac{\sin u - u\cos u}{u^3} \cdot \cos u/2 \cdot du$$

Changing u to x $\int_0^\infty \frac{[-\sin x + x\cos x]}{x^3} \cos(x/2)\, dx = 3\pi/16$

$$\int_0^{\infty} \frac{x\cos x - \sin x}{x^3}\, dx = -\frac{3\pi}{16}$$

(3) Find the fourier transform of $f(x) = e^{-|x|}$

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$\left\{ |x| = \begin{array}{ll} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{array} \right.$

Sol: Fourier transform of $f(x)$ is given by $F(u) = \int_{-\infty}^{\infty} f(x) e^{iux}\, dx$

here $f(x) = \begin{cases} e^{-x} & \text{for } x \geq 0 \\ e^{x} & \text{for } x < 0 \end{cases}$

$$F(u) = \int_{-\infty}^{0} e^{x} \cdot e^{iux}\, dx + \int_0^{\infty} e^{-x} e^{iux}\, dx$$

$$= \int_{-\infty}^{0} e^{(1+iu)x}\, dx + \int_0^{\infty} e^{-(1-iu)x}\, dx$$

$$= \left[\frac{e^{(1+iu)x}}{1+iu} \right]_{-\infty}^{0} + \left[\frac{e^{-(1-iu)x}}{-(1-iu)} \right]_0^{\infty}$$

$$= \left\{ \frac{1}{1+iu} - 0 \right\} + \left\{ 0 - (-) \frac{1}{1-iu} \right\}$$

$$= \frac{1}{1+iu} + \frac{1}{1-iu}$$

$$= \frac{1-iu+1+iu}{(1+iu)(1-iu)}$$

$$= \frac{+2}{1+i^2u^2}$$

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$$F(u) = \frac{2}{1+u^2}$$

(4) June 2018 Find the fourier transform of $f(x) = \begin{cases} 1-|x| & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$ & hence deduce

that $\int_0^{\infty} \frac{\sin^2 t}{t^2}\, dt = \pi/2$

Solⁿ:

$$|x| = x \text{ for } x \geq 0$$
$$= -x \text{ for } x < 0$$

$$f(x) = \begin{cases} 1-|x| & \text{for } -1 \le x \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

$$F(u) = \int_{-1}^{1} \{1-|x|\} e^{iux}\, dx$$

$$F(u) = \int_{-1}^{0} \{1-(-x)\} e^{iux}\, dx + \int_0^1 \{1-(x)\} e^{iux}\, dx$$

$$= \int_{-1}^{0} (1+x) e^{iux}\, dx + \int_0^1 (1-x) e^{iux}\, dx$$

A.B.R

$$= \left[(1+x)\frac{e^{iux}}{iu} - (1)\frac{e^{iux}}{i^2u^2} \right]_{-1}^{0} + \left[(1-x)\frac{e^{iux}}{iu} - (-1)\frac{e^{iux}}{i^2u^2} \right]_0^1$$

$$= \left[\left\{\frac{1}{iu} + \frac{1}{u^2}\right\} - \left\{0 + \frac{e^{-iu}}{u^2}\right\}\right] + \left[\left\{0 + \frac{e^{iu}}{-u^2}\right\} - \left\{\frac{+1}{iu} + \frac{1}{-u^2}\right\}\right]$$

$$= \cancel{\frac{-i}{u}} + \frac{1}{u^2} - \frac{e^{-iu}}{u^2} - \frac{e^{iu}}{u^2} + \cancel{\frac{i}{u}} + \frac{1}{u^2}$$

$$= \frac{2}{u^2} - \frac{1}{u^2}\left[e^{-iu} + e^{iu}\right]$$

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$= \frac{2}{u^2} - \frac{1}{u^2}\left[\{\cos u - i\sin u\} + \{\cos u + i\sin u\}\right]$

$= \frac{2}{u^2} - \frac{2\cos u}{u^2}$

$= \frac{2}{u^2}(1-\cos u)$

w.k.T

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos\theta = 2\cos^2\theta/2 - 1$

$\cos\theta = 1 - 2\sin^2\theta/2$

$2\sin^2\theta/2 = 1 - \cos\theta$

$= \frac{2}{u^2} \times 2\sin^2 u/2$

$F(u) = \frac{4\sin^2 u/2}{u^2}$

by Inverse F.T

$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4\sin^2 u/2}{u^2}\, e^{-iux}\, du$ — (*)

(*) >

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin^2 u/2}{u^2/4} \cdot e^{-iux}\, du$$

put $x=0$ $f(0)=1$

$$1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin^2 u/2}{(u/2)^2}\, du$$

put $u/2 = t \Rightarrow du = 2dt$

u varies from $-\infty$ to ∞

t varies from $-\infty$ to ∞

$$1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2} \cdot 2 \cdot dt$$

$\frac{\sin^2 t}{t^2}$ is even function

$$I = \frac{1}{\pi} 2 \int_0^{\infty} \frac{\sin^2 t}{t^2}\, dt$$

$$\int_0^{\infty} \frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2}$$

(5) Find the fourier Sine & cosine transforms of $f(x) = e^{-\alpha x}$, $\alpha > 0$

June 2018

Soln: fourier Sine and Cosine transforms are given by

$$F_S(u) = \int_0^{\infty} f(x) \sin ux \, dx \text{ and } F_C(u) = \int_0^{\infty} f(x) \cos ux \, dx$$

$$F_S(u) = \int_0^{\infty} e^{-\alpha x} \cdot \sin ux \, dx$$

W.K.T $\int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2} \left[a \sin bx - b \cos bx \right]$

$$= \left\{ \frac{e^{-\alpha x}}{(-\alpha)^2 + u^2} \left[-\alpha \sin ux - u \cos ux \right] \right\}_0^{\infty}$$

$e^{-\alpha x} \to 0$ as $x \to \infty$ i.e $e^{-\infty} = 0$

$e^{0} = 1$

$$= 0 - \frac{1}{\alpha^2 + u^2} \left[-\alpha \sin(0) - u \cos(0) \right]$$

$$= \frac{-1}{\alpha^2 + u^2} \left[0 - u \right] = \frac{u}{\alpha^2 + u^2}$$

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$$F_s(u) = \frac{u}{\alpha^2+u^2}$$

$$F_c(u) = \int_0^\infty f(x)\cos ux \, dx$$

$$= \int_0^\infty e^{-\alpha x} \cos ux \, dx$$

w.k.t $\int e^{ax}\cos bx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

$$= \left[\frac{e^{-\alpha x}}{(-\alpha)^2+u^2}\left[-\alpha\cos ux + u\sin ux\right)\right]_0^\infty$$

$e^{-\infty} = 0$

$$= 0 - \frac{1}{\alpha^2+u^2}\left(-\alpha\cos(0) + u\sin(0)\right)$$

$$= \frac{-1}{\alpha^2+u^2} \times -\alpha(1)$$

$$F_c(u) = \frac{\alpha}{\alpha^2+u^2}$$

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⑥ Obtain the fourier Cosine tranyform of the function

$$f(x) = \begin{cases} 4x, & 0 < x < 1 \\ 4-x, & 1 < x < 4 \\ 0, & x > 4 \end{cases}$$

Solⁿ: Fourier Cosine transform is given by

$$F_c(u) = \int_0^{\infty} f(x)\cos ux \, dx$$

$$= \int_0^1 f(x)\cos ux\,dx + \int_1^4 f(x)\cos ux\,dx + \int_4^{\infty} f(x)\cos ux\,dx$$

$$= \int_0^1 4x\cos ux\,dx + \int_1^4 (4-x)\cos ux\,dx + \int_4^{\infty} 0\cdot\cos ux\,dx$$

$$= \int_0^1 4x\cos ux\,dx + \int_1^4 (4-x)\cos ux\,dx + 0$$

Apply B. Rule

$$= \left[4x \times \frac{\sin ux}{u} + 4\cdot\frac{\cos ux}{u^2}\right]_0^1 + \left[(4-x)\frac{\sin ux}{u} - (-1)\times -c\right.$$

$$= \left[\left\{4\frac{\sin u}{u} + \frac{4\cos u}{u^2}\right\} - \left\{0 + \frac{4}{u^2}\right\}\right] + \left[\left\{0 - \frac{\cos 4u}{u^2}\right\} - \left\{\right.\right.$$

$$= \frac{4\sin u}{u} - \frac{3\sin u}{u} + \frac{4\cos u}{u^2} + \frac{\cos u}{u^2} - \frac{4}{u^2} - \frac{\cos 4u}{u^2}$$

$$= \frac{\sin u}{u} + 5\frac{\cos u}{u^2} - \frac{1}{u^2}(4 + \cos 4u)$$

OR

$$= \frac{\sin u}{u} + \frac{5\cos u}{u^2} - \frac{4}{u^2} - \frac{\cos 4u}{u^2}$$

OR

$$F_c(u) = \frac{\sin u}{u} + \frac{5\cos u - 4 - \cos 4u}{u^2}$$

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Do yourself

(7) Find the fourier cosine transform of

$$f(x) = \begin{cases} x & \text{for } 0<x<1 \\ 2-x & \text{for } 1<x<2 \\ 0 & \text{for } x>2 \end{cases}$$

Sol:

$$F_c(u) = \int_0^\infty f(x)\cos ux \, dx$$

$$= \int_0^1 f(x)\cos ux\,dx + \int_1^2 f(x)\cos ux\,dx + \int_2^\infty f(x)\cos ux\,dx$$

$$= \int_0^1 x\cos ux\,dx + \int_1^2 (2-x)\cos ux\,dx + \int_2^\infty 0\cdot\cos ux\,dx$$

$$= \int_0^1 x\cos ux\,dx + \int_1^2 (2-x)\cos ux\,dx + 0$$

$$= \left[x\frac{\sin ux}{u} + \frac{\cos ux}{u^2}\right]_0^1 + \left[(2-x)\frac{\sin ux}{u} - (-1)\times -\frac{\cos ux}{u^2}\right]_1^2$$

$$= \left\{\left(\frac{\sin u}{u} + \frac{\cos u}{u^2}\right) - \left(0 + \frac{1}{u^2}\right)\right\} + \left\{\left(0 - \frac{\cos 2u}{u^2}\right) - \left(\frac{\sin u}{u} - \frac{\cos u}{u^2}\right)\right\}$$

$$= \cancel{\frac{\sin u}{u}} + \frac{\cos u}{u^2} - \frac{1}{u^2} - \frac{\cos 2u}{u^2} - \cancel{\frac{\sin u}{u}} + \frac{\cos u}{u^2}$$

$$= \frac{2\cos u}{u^2} - \frac{1}{u^2} - \frac{\cos 2u}{u^2}$$

$$F_c(u) = \frac{2\cos u - \cos 2u - 1}{u^2}$$

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(8) Find the fourier Sine transform of

Dec 2018 $f(x) = e^{-|x|}$ & hence evaluate

$$\int_0^\infty \frac{x \sin mx}{1+x^2}\, dx \quad m>0$$

Soln: Fourier Sine transform is given by

$$F_s(u) = \int_0^\infty f(x) \sin ux \, dx$$

$$F_s(u) = \int_0^\infty e^{-|x|} \sin ux \, dx$$

$$= \int_0^\infty e^{-x} \sin ux \, dx \quad \text{since } |x| = x,\ x>0$$

W.K.T $\int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx)$

$$= \left[\frac{e^{-x}}{(-1)^2+u^2}\left\{-\sin ux - u\cos ux\right\}\right]_0^\infty$$

$$= \left[\frac{e^{-x}}{1^2+u^2}\left(-\sin ux - u\cos ux\right)\right]_0^\infty$$

$\Rightarrow$ $e^{-\infty} = 0,\ e^0 = 1,\ \sin 0 = 0$
$\cos 0 = 1$

$$= 0 - \frac{1}{1+u^2}(0-u)$$

$$F_s(u) = \frac{u}{1+u^2}$$

By inverse fourier Sine transform we have

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(u) \sin ux \, du$$

put $x = m$ $\quad f(x) = e^{-|m|} = e^{-m}$

$$e^{-m} \cdot \frac{\pi}{2} = \int_0^\infty \frac{u}{1+u^2} \sin um \cdot du$$

$$\int_0^\infty \frac{u \sin mu}{1+u^2} du = \frac{\pi}{2} e^{-m}$$

changing u to x

$$\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}$$

(9) Find the fourier Sine transform of $\frac{e^{-ax}}{x}$, $a > 0$

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Soln: $F_s(u) = \int_0^\infty f(x) \sin ux \, dx$

$$F_s(u) = \int_0^\infty \frac{e^{-ax}}{x} \sin ux \, dx \quad \text{—— (1)}$$

we cannot evaluate this integral directly and hence we proceed as follows

$$\frac{d}{du}\left[F_s(u)\right] = \int_0^\infty \frac{e^{-ax}}{x} \frac{\partial}{\partial u}(\sin ux) \, dx$$

$$= \int_0^\infty \frac{e^{-ax}}{\cancel{x}} \cdot \cos ux \cdot \cancel{x} \, dx$$

$$F_s(u) = \int_0^\infty e^{-ax} \cos ux \, dx \quad — \circledast$$

$$= \left[\frac{e^{-ax}}{(-a)^2+u^2} (-a\cos ux + u\sin ux) \right]_{x=0}^{\infty}$$

$$= 0 - \frac{1}{a^2+u^2} (-a+0)$$

$$\frac{d}{du}\left[F_s(u)\right] = \frac{a}{a^2+u^2}$$

by integrating w.r.to u on B.S we get

$$\int \frac{d}{du} F_s(u) \, . \, du = \int \frac{a}{a^2+u^2} du$$

$$F_s(u) = \tan^{-1}(u/a) + C$$

To find C, put $u=0$

$$F_s(0) = \tan^{-1}(0) + C$$

$F_s(0) = 0$ from (1)

$$\underline{F_s(u) = \tan^{-1} u/a}$$

(10) Dec 2016 Find the inverse Fourier sine transform of $\hat{f}_s(\alpha) = \frac{1}{\alpha} e^{-a\alpha}$, $a > 0$

By data $\hat{f}_s(\alpha) = \frac{1}{\alpha} e^{-a\alpha}$

Soln :

$$F_s[f(x)] = \hat{f}_s(\alpha) = \int_0^\infty f(x) \sin \alpha x \, dx$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \hat{f}_s(\alpha) \sin \alpha x \, d\alpha \quad — (1)$$

$$= \frac{2}{\pi}\int_0^\infty \frac{e^{-\alpha a}}{\alpha} \sin \alpha x \, d\alpha$$

D.w.r. to x

$$\frac{d}{dx}[f(x)] = \frac{2}{\pi}\int_0^\infty \frac{d}{d}\left(\frac{e^{-\alpha a}}{\alpha}\frac{\partial}{\partial x}(\sin \alpha x)\right) d\alpha$$

$$= \frac{2}{\pi}\int_0^\infty \left[\frac{e^{-\alpha a}}{\cancel{\alpha}} \times \cos \alpha x \cdot \cancel{\alpha}\right] d\alpha$$

$$= \frac{2}{\pi}\int_0^\infty e^{-\alpha a} \cos \alpha x \cdot d\alpha$$

$$= \frac{2}{\pi}\left[\frac{e^{-a\alpha}}{a^2 + x^2}\left(-a\cos \alpha x + x \sin \alpha x\right)\right]_0^\infty$$

$$= \frac{2}{\pi}\left[0 - \frac{1}{a^2+x^2}(-a+0)\right]$$

$$\frac{d}{dx} f(x) = \frac{2}{\pi} \times \frac{a}{a^2+x^2}$$

Int. w.r. to x

$$f(x) = \frac{2}{\pi}\int_0 \frac{a}{a^2+x^2} dx$$

$\tan^{-1} x/a = \frac{1}{1+x^2/a^2}$

$=$

$$f(x) = \frac{2}{\pi}\tan^{-1}\frac{x}{a} + C \quad — (*)$$

To find C, put $x = 0$ in $(*)$

$$f(0) = 0 + C$$

$$0 = 0 + C \Rightarrow \underline{C = 0}$$

$$f(x) = 2/\pi \tan^{-1} x/a$$

(*) Find the fourier Cosine transform of

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$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2-x & \text{for } 1 < x < 2 \\ 0 & \text{for } x > 2 \end{cases}$$

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(17) Find the infinite fourier Cosine transform of e^{-x^2}

Sol:

Fourier cosine transform is given by

$$F_c(u) = \int_0^\infty f(x) \cos ux \, dx$$

$$F_c(u) = \int_0^\infty e^{-x^2} \cos ux \, dx$$

we cannot evaluate the integral directly & hence proceed as follows. The process is called diffⁿ under the integral sign.

$$\frac{dF_c}{du} = \int_0^\infty \frac{\partial}{\partial u} \left(e^{-x^2} \cos ux\right) dx$$

$$= \int_0^\infty e^{-x^2} (-\sin ux \cdot x) \, dx$$

$$= \frac{1}{2} \int_0^\infty \sin ux \left\{ e^{-x^2} (-2x) \right\} dx$$

$$2 \frac{dF_c}{du} = \int_0^\infty \sin ux \left\{ e^{-x^2} (-2x) \right\} dx$$

Integrate RHS by parts we have

$$= \left[\sin ux \left(e^{-x^2} \right) \right]_0^\infty - \int_0^\infty e^{-x^2} (\cos ux \cdot u) \, dx$$